Section 11.2: Series

Objective: In this lesson, you learn how to

- ☐ define a series and determine its convergence or divergence using partial sums and analyze geometric series, as well as harmonic series.

I. Series

Definition: Infinite series or series

An **infinite series or series** is the sum of an infinite sequence $a_1 + a_2 + a_3 + \cdots$ and is denoted by

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum a_n.$$

Definition: Partial sums

If $\sum_{n=1}^{\infty} a_n$ is a series, then

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

is called its n^{th} **Partial sum**.

Remark: s_n is the partial sum of terms in the sequence $\{a_n\}$ from 1 to n, therefore,

$s_1 = a_1,$

$s_2 = a_1 + a_2 = s_1 + a_2,$

$s_3 = a_1 + a_2 + a_3 = s_2 + a_3,$

$s_4 = a_1 + a_2 + a_3 + a_4 = s_3 + a_4.$ etc....

Example 1: Find the first five partial sum terms of $\sum_{i=1}^{n} i$

$S_1 = 1$

$S_2 = 1 + 2 = 3$

$S_3 = 1 + 2 + 3 = S_2 + 3 = 3 + 3 = 6$

$S_4 = 1 + 2 + 3 + 4 = 6 + 4 = 10$

$S_5 = 1 + 2 + 3 + 4 + 5 = 10 + 5 = 15$

Convergent and divergent

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n^{th} partial sum $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

- The series $\sum a_n$ converges if the sequence of partial sums $\{s_n\}$ is convergent and we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i = s,$$

  the number s is called **the sum of the series**.

- The series $\sum a_n$ diverges if the sequence of partial sums $\{s_n\}$ is divergent (i.e. $\lim_{n\to\infty} s_n =$ DNE).

$$\overbrace{1+2+3+4+5+6}^{7}$$

$$7$$

$$7 \cdot \frac{6}{2}$$

Example 2: Is the series $\sum_{i=1}^{\infty} i$ convergent or divergent?

$$s_n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty \text{ div}$$

so, $\sum_{i=1}^{\infty} i$ is divergent

How To Shift a Series:

Example 3: Adjust the series

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+1}}{n-3}, = \frac{(-1)^5}{1} + \frac{(-1)^6}{2} + \frac{(-1)^7}{3} + \cdots$$

so that the index will now start at $n = 1$.

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+1 \to n+3}}{n-3 \to n+3} = \sum_{n=1}^{\infty} \frac{(-1)^{n+4}}{n} = \frac{(-1)^5}{1} + \frac{(-1)^6}{2} + \frac{(-1)^7}{3} + \cdots$$

$$\frac{n=4 \quad -3}{n=1}$$

Definition: Geometric series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}, \ a \neq 0,$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

if $|r| \geq 1$, the geometric series is divergent

Proof : Consider the partial sums:

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + \cancel{ar^n}$$

$$- \quad rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + ar^4 + \cdots + \cancel{ar^n} + ar^{n+1}$$

$$S_n - rS_n = a - ar^{n+1}$$

$$S_n(1-r) = a(1-r^{n+1})$$

$$\boxed{S_n = \frac{a(1-r^{n+1})}{(1-r)}} \qquad r \neq 1$$

$$-1 < r < 1$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} S_n = \begin{cases} \dfrac{a}{1-r} & |r|<1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$$

if $|r| < 1$ $\quad \lim_{n\to\infty} r^{n+1} =$

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^{n+1} = \lim_{n\to\infty} \frac{1}{2^{n+1}} = \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0$$

$$|r| \geq 1 \Rightarrow \lim_{n\to\infty} \frac{a(1-3^{n+1})}{1-3} = \frac{-a}{2} \lim_{n\to\infty} (1-3^{n+1})$$

$$= \frac{-a}{2}(1-3^{\infty})$$

$$= \frac{-a}{2}(1-\infty)$$

$$= \frac{-a}{2}(-\infty) = \infty$$

Example 4: Present the number $6.2828282\cdots = 6.\overline{28}$ as a ratio of integers.

$$6.28282828\ldots = 6 + .28 + 0.0028 + 0.000028 + 0.00000028 + \cdots$$
$$= 6 + \frac{28}{100} + \frac{28}{10000} + \frac{28}{1000000} + \frac{28}{100000000} + \cdots$$
$$= 6 + \frac{28}{100^1} + \frac{28}{(100)^2} + \frac{28}{(100)^3} + \frac{28}{(100)^4} + \cdots$$
$$= 6 + \sum_{n=1}^{\infty} 28\left(\frac{1}{100}\right)^n = 6 + \sum_{n=1}^{\infty} 28 * \frac{1}{100}\left(\frac{1}{100}\right)^{n-1}$$

$a = \frac{28}{100}$, $r = \frac{1}{100}$, Now $\left|\frac{1}{100}\right| < 1$

So, the Geometric series is convergent, and the sum

$$= 6 + \sum_{n=1}^{\infty} 28\left(\frac{1}{100}\right)^n = 6 + \frac{a}{1-r} = 6 + \frac{28/100}{1-1/100}$$
$$= 6 + \frac{28/\cancel{100}}{99/\cancel{100}}$$
$$= 6 + \frac{28}{99} = \frac{6*99+28}{99}$$
$$= \frac{622}{99}$$

Example 5: Is the following series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

$$\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 3^1 \cdot 3^{-n} = \sum_{n=1}^{\infty} 4^n \cdot 3 \cdot \frac{1}{3^n}$$

$$(a^b)^c = a^{bc}$$

$$= \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n = \sum_{n=1}^{\infty} \cancel{3} \cdot \frac{4}{\cancel{3}}\left(\frac{4}{3}\right)^{n-1}$$
$$= \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

$$a^n = a^{n-1} a^1$$

$r = \frac{4}{3}$

Geo. series $|r| = \left|\frac{4}{3}\right| > 1 \Rightarrow$ the series is divergent.

$$\sum_{r=1}^{\infty} a r^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

$$\frac{ar}{a} = r$$

$$x\, x^2 = x^3$$
$$x^3 = x^{3-1} \cdot x^1$$

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = 2^2 \cdot 3^{1-1} + 2^4 \cdot 3^{1-2} + \cdots$$

$a = 4$ $\quad r = \frac{2^{\cancel{4}\,2} \cdot 3^{-1}}{\cancel{2^2}} = \frac{2^2}{3} = \frac{4}{3}$

$$\lim_{n\to\infty} \frac{1}{2^n} = 0$$

Example 6: Find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \overset{a}{\frac{1}{2}} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$r = \frac{1/4}{1/2} = 1/2$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \underset{a}{\frac{1}{2}} \underset{r}{\left(\frac{1}{2}\right)}^{n-1}$$

$|r| = \left|\frac{1}{2}\right| < 1$ conv,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

Example 7: Find the value(s) of c so that

$$\sum_{n=1}^{\infty} (2c-5)^n$$

will converge. Find the sum for those values of c.

$|x| > 1$

$x < -1$ | $|x| < 1$ | $x > 1$

-1 0 1

$$\left|\frac{-1}{2}\right| = \frac{1}{2} < 1$$

$$\sum_{n=1}^{\infty} (2c-5)^n = \sum_{n=1}^{\infty} (2c-5)\underset{r}{(2c-5)}^{n-1}$$

the series convergent if $|r| < 1$

$$|2c-5| < 1$$

$|x| < 1$

$-1 < x < 1$

$|x| > 1$

$x < -1$ or $x > 1$

$$-1 < 2c-5 < 1$$
$$+5 \quad +5 \quad +5$$
$$\frac{4}{2} < \frac{2c}{2} < \frac{6}{2} \Rightarrow \boxed{2 < c < 3}$$

The sum if $2 < c < 3$, then $a = 2c-5$, $r = 2c-5$

$$\text{The sum} = \frac{a}{1-r} = \frac{2c-5}{1-(2c-5)} = \frac{2c-5}{-2c+6}$$

For example: $c = 21/10$

$$\text{The sum} = \frac{2\cdot\left(\frac{21}{10}\right)-5}{-2\left(\frac{21}{10}\right)+6}$$

Definition: Telescoping series

A **telescoping series** is a series where each term b_n can be written as

$$b_n = a_n - a_{n+1},$$

for some series b_n.

Notice: To find the sum, we have

$$S_n = b_1 + b_2 + b_3 + \ldots + b_n = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \ldots + (a_n - a_{n+1}) = a_1 - a_{n+1}.$$

Therefore,

$$S_n = a_1 - a_{n+1}.$$

Thus, the telescoping series is convergent if $a_{n+1} \longrightarrow$ **finite number**, and the sum is

$$S = \lim_{n\to\infty} S_n = a_1 - \lim_{n\to\infty} a_{n+1}$$

$$\lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = 0$$

Example 8: Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

is convergent and find its sum.

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} \qquad n=-1 \qquad n=-2$$

$$A = \frac{1}{-1+2} = 1$$

$$B = \frac{1}{-2+1} = -1$$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

~~Try this~~

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) ?$$

$$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$S_n = \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cancel{\ldots} + \left(\cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}\right) + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)$$

(1, 2, 3, n-1, n)

$$S_n = \frac{1}{2} - \frac{1}{n+2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}$$

0

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2}$$

The series is convergent and the sum is 1/2.

Example 9: Show that the **harmonic**

$\lim_{n\to\infty} \frac{1}{n} = 0$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is divergent.

proof: consider the partial sums

$S_2, S_4, S_8, S_{16}, \ldots$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$

$S_{16} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{16} > \cdots = 1 + \frac{4}{2}$

$\vdots$

$S_{2^n} > 1 + \frac{n}{2}$

$S = \lim_{n\to\infty} S_{2^n} > \lim_{n\to\infty} 1 + \frac{n}{2} = \infty$

therefore, S_n does not converge

so $\sum_{n=1}^{\infty} \frac{1}{n}$ div.

Theorem 1

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Note: The converse is not true ingeneral.

$$\text{if } \lim_{n\to\infty} a_n = 0, \textbf{ we cannnot conclude that } \sum a_n \text{ is convergent}$$

Test for Divergence. TD

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Remark:

- The statement follows from the theorem immediately preceding it, since if the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$.
- Note that if $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ **may converge or it may diverge.**

Example 12: Determine whether the series is convergent or divergent. If it is convergent, find it sum

a. $\sum_{n=1}^{\infty} \dfrac{n-1}{3n-1}$

$a_n = \dfrac{n-1}{3n-1}$

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots b_0} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \frac{a_n}{b_m} & n=m \\ 0 & m>n \\ \infty & n>m \end{cases}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{3n-1} = \lim_{n\to\infty} \frac{\cancel{n}}{3\cancel{n}} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} \neq 0$$

then $\sum_{n=1}^{\infty} \dfrac{n-1}{3n-1}$ is divergent by TD

b. $\sum_{n=1}^{\infty} n \sin\left(\dfrac{1}{n}\right)$

$\infty \cdot 0$

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\frac{1}{\infty}\right) = \sin(0) = 0$$

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \quad \frac{0}{0}$$

derivative

derivative

$$= \lim_{n\to\infty} \frac{\cos(1/n)\cancel{(-1/n^2)}}{\cancel{(-1/n^2)}}$$

$$\frac{1}{\frac{1}{n}} = n$$

$$= \lim_{n\to\infty} \cos(1/n) = \cos(0) = 1 \neq 0$$

L'H Rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \frac{0}{0} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Then $\sum_{n=1}^{\infty} n \sin(1/n)$ is divergent by TD

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

c. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

$$\lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \ln(1) = 0$$

Test failed.

$$S_n = \sum_{n=1}^{n} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{n} \ln(n) - \ln(n+1)$$

$$= (\cancel{\ln(1)}^{0} - \cancel{\ln(2)}) + (\cancel{\ln 2} - \cancel{\ln 3}) + (\cancel{\ln(3)} - \cancel{\ln 4}) + \cancel{\cdots}$$

$$+ (\ln(n-1) - \cancel{\ln(n)}) + (\cancel{\ln(n)} - \ln(n+1))$$

$$S_n = -\ln(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\ln\left(\lim_{n\to\infty} n+1\right)$$

$$= -\ln(\infty) = -\infty \quad \text{div.}$$

Theorem.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum_{n=1}^{\infty} ca_n$ (where c is a constant), $\sum_{n=1}^{\infty}(a_n + b_n)$, and $\sum_{n=1}^{\infty}(a_n - b_n)$, and

a. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$.

b. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

c. $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Example 13: Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{2^n} - \frac{26}{(n+1)(n+2)}\right)$?

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = \sum_{n=1}^{\infty} 5 \cdot \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \rightarrow a = \frac{5}{2},\ r = \frac{1}{2} \rightarrow |r| < 1$$

conv.

$$\sum_{n=1}^{\infty} \frac{5}{2^n} = \frac{5}{2} + \frac{5}{2^2} \rightarrow r = \frac{5/2^2}{5/2} = 1/2$$

a ar

Geo ser. $r = 1/2$, $a = 5/2 \longrightarrow$ sum $= \dfrac{a}{1-r} = \dfrac{5/2}{1-1/2}$

$= \boxed{5}$

$$\sum_{n=1}^{\infty} \frac{26}{(n+1)(n+2)} = 26 \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$= 26 \cdot \frac{1}{2} = 13$$

from Example 8

$$\sum_{n=1}^{\infty} \frac{5}{2^n} - \frac{26}{(n+1)(n+2)} = 5 - 13 = -8$$